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One-loop mean-field theory for lattice Abelian Higgs model

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Abstract. We consider the lattice Abelian Higgs model with frozen radial degrees of freedom using the mean-field approximation with one-loop corrections. In the weak-coupling region the behaviour of the frequencies arising in the expansion of the action allows the Higgs and Coulomb phases to be distinguished. Analytical results are presented for all phase transition lines. The phase structure obtained is in qualitative agreement with Monte Carlo calculations for Higgs charges $q = 1$, 2 and 6.

1. Introduction

The mean-field theory including corrections (Drouffe 1980, Brezin and Drouffe 1982, Drouffe and Zuber 1983) can be considered as the natural weak-coupling perturbation theory for lattice gauge theories. Using the mean-field technique for a determination of the phase structure of the lattice Abelian Higgs model we find that the tree approximation does not reproduce the phase structure predicted from Monte Carlo simulations (Ranft *et al* 1983). In particular, working without gauge-fixing for a Higgs charge $q = 1$ no analytical connection between confinement and Higgs phases appears and for $q = 6$ the Coulomb phase is absent for increasing Higgs coupling.

It was shown by Drouffe (1980) that the lowest-order mean-field equations correspond to a saddle-point approximation which allows the calculation of loop corrections (Brezin *et al* 1976). It has also been proved that the mean-field approach can be reconciled with Elitzur's theorem (1975) in this way. An improvement of the precision of the mean-field calculations can be achieved including corrections beyond the one-loop approximation (Flyvbjerg *et al* 1983, Flyvbjerg 1984). However, the one-loop corrections have been successfully applied to several models (see, for instance, Alessandrini *et al* 1982, Alberty *et al* 1983, Alessandrini 1983, Alessandrini and Boucaud 1983, Dagotto 1983, Trinchero 1983, Boucard 1984). Therefore it should be sufficient to include only one-loop corrections for a reproduction of the true phase structure.

We have determined the phase structure of the lattice Abelian Higgs model using the mean-field approximation on the one-loop level. Considering different Higgs charges $q = 1$, 2 and 6 we obtain qualitative agreement with Monte Carlo calculations (Ranft *et al* 1983). Furthermore we present analytical approximations for all phase transition lines.

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2. The effective action for the lattice Abelian Higgs model

The theory is defined in a d-dimensional lattice of N sites. The partition function is given by

$$Z(\beta,\kappa)=\int_{U(1)} \mathrm{d}\mu[U]\,\mathrm{d}\mu[\sigma]\exp\Big(\beta\sum_{\text{plaquettes}}U_p+\kappa\sum_{\text{Links}}(\sigma U^q\sigma)_L\Big). \tag{2.1}$$

The $U_L=\exp(\mathrm{i}\Theta_L)$ are the link variables of the gauge fields and $\sigma_x=\exp(\mathrm{i}\chi_x)$ the site variables of the Higgs matter fields for frozen radial modes. The Θ_L and χ_x are angle variables. The power q is the Higgs charge. We rewrite the partition function using the standard procedure (Brezin and Drouffe 1982) and obtain

$$\begin{aligned} Z(\beta,\kappa)=\int_{\text{complex plane}} &\prod_x \mathrm{d}H_x\frac{\mathrm{d}B_x\,\mathrm{d}B_x^*}{(4\pi)^2}\prod_L \mathrm{d}V_L\frac{\mathrm{d}A_L\,\mathrm{d}A_L^*}{(4\Pi)^2}\prod_L \mathrm{d}C_L\,\mathrm{d}W_L \\ &\times\delta(C+2\mathrm{i}\kappa HH^*)\delta(W-V^q)\exp(S_{\text{eff}}). \end{aligned} \tag{2.2}$$

After the integration the auxiliary fields C_L and W_L satisfy

$$C_L=-\mathrm{i}2\kappa H_xH^*_{x+\mu}\qquad W_L=V_L^q \tag{2.3}$$

where μ is a lattice unit vector and $L\equiv(x,\mu)$. The effective action S_{eff} is defined by

$$\begin{aligned} S_{\text{eff}}=\beta\sum_p V_p+\kappa\sum_L(HWH)_L+\sum_L[-\tfrac{1}{2}\mathrm{i}(A_L^*V_L+C_L^*W_L+\text{cc}) \\ +\ln f_L^{(q)}(A,C)]+\sum_x[-\tfrac{1}{2}\mathrm{i}(B_x^*H_x+\text{cc})+\ln f_x(B)]. \end{aligned} \tag{2.4}$$

The one-link integral $f_L^{(q)}(A,C)$ and the one-site integral $f_x(B)$ are

$$f_L^{(q)}(A,C)=\int_0^{2\pi}\frac{\mathrm{d}\Theta_L}{2\pi}\exp[\tfrac{1}{2}\mathrm{i}(A_L^*U_L+C_L^*U_L^q+\text{HC})] \tag{2.5a}$$

$$f_x(B)=\int_0^{2\pi}\frac{d\chi_x}{2\pi}\exp[\tfrac{1}{2}\mathrm{i}(B_x^*\sigma_x+\text{HC})]. \tag{2.5b}$$

The C_L integration leads to a cancalation of the $\kappa(HWH)_L$ terms and the (C_LW_L) contributions in the effective action. Then S_{eff} depends on H_x only via the one-link integral and the (B_xH_x) terms. We call V_L and H_x the effective gauge and Higgs fields respectively.

3. The mean-field approximation including corrections

In the tree approximation one performs a saddle-point evaluation of equation (2.2). Looking for translationary invariant solutions $(V_L, H_x, A_L, B_x)=(m, M, -\mathrm{i}Z_1, -\mathrm{i}Z_3)$ and fixing the gauge so that m, M, Z_1, Z_3 are real numbers we obtain the following saddle-point equations

$$\begin{aligned} &m=(\partial/\partial Z_1)\ln f_L^{(q)}(A_L=-\mathrm{i}Z_1, C_L=-\mathrm{i}Z_2)\equiv(\partial/\partial Z_1)\ln f^{(q)}(Z_1,Z_2) \\ &M=(\mathrm{d}/\mathrm{d}Z_3)\ln f_x(B_x=-\mathrm{i}Z_3)\equiv(\mathrm{d}/\mathrm{d}z_3)\ln I_0(z_3) \\ &Z_1=4\bar\beta m^3\qquad Z_2=2\kappa M^2\qquad Z_3=4d\kappa m^qM\qquad \bar\beta=\beta(d-1) \end{aligned} \tag{3.1}$$

where $I_n(Z)$ are modified Bessel functions. There are three types of leading solutions (m, M) of equation (3.1). For each of them we calculate the free energy per link, f_0. On the level of the tree approximation we relate the three types of solutions to the phases of the model. A crossing of the free energies is interpreted as phase transition. The mean values (m, M), the free energies f_0 and the corresponding phases are:

$$\begin{array}{ll}
(0,0) & f_0^{\text{conf}}=0 \\
 & \text{confinement phase (strong coupling)} \\
\left(1-\dfrac{1}{8\bar\beta},0\right) & f_0^{\text{coul}}=-\bar\beta m^4+Z_1 m-\ln I_0(Z_1) \\
 & \text{Coulomb phase (weak coupling, } \bar\beta \text{ large, } \kappa \text{ small)} \\
\left(1-\dfrac{1}{8\bar\beta+4q^2\kappa},1-\dfrac{1}{8d\kappa}\right) & f_0^{\text{Higgs}}=-\bar\beta m^4+Z_1 m-\ln f^{(q)}(Z_1,Z_2) \\
 & \qquad +(1/d)[Z_3 M-\ln I_0(Z_3)] \\
 & \text{Higgs phase (weak coupling, } \bar\beta \text{ large, } \kappa \text{ large).}
\end{array} \tag{3.2}$$

The non-trivial mean values m and M are approximations. Ranft *et al* (1983) have predicted that the Higgs mean value $M(\kappa)$ vanishes in a square-root-type fashion going from the Higgs to the Coulomb phases. The approximation (3.2) seems to be in contradiction to this, but in fact, this is not so (see appendix 2).

If the gauge is not fixed the non-trivial saddle-points are degenerated. For a determination of the phase structure on the tree level, however, we can use the mean values (3.1) due to the invariance of the free energies f_0. Recently it was shown (Flyvbjerg *et al* 1983, Flyvbjerg 1984) that the axial gauge gives very good results using the mean-field technique. However, our model has a Coulomb phase characterised by a U(1) gauge symmetry. In this case the theorem of Elitzur (1975) predicts a vanishing expectation value for the fields. To reconcile the mean-field approach with this prediction one needs to include the gauge degrees of freedom. Therefore we follow Alessandrini *et al* (1982) using collective coordinate methods (Gervais and Sakita 1975, Polyakov 1977) and consider the gauge degrees of freedom calculating the one-loop corrections.

To improve the tree approximation we perform a loop expansion around the saddle points. Considering small quantum fluctuations

$$\begin{array}{ll}
V_L=m+\lambda_L+\mathrm{i}\xi_L & A_L=-\mathrm{i}Z_1+X_L+\mathrm{i}Y_L \\
H_x=M+\alpha_x+\mathrm{i}\gamma_x & B_x=-\mathrm{i}Z_3+n_x+\mathrm{i}P_x
\end{array} \tag{3.3}$$

we expand the effective action in terms of these fluctuations up to the second order (linear terms vanish). We calculate the corrections Δf to the free energy $f=f_0+\Delta f$ after integrating out the fluctuations in the partition function.

For the trivial saddle point (0, 0) this treatment gives the usual strong-coupling expansion. Considering only the zeroth order we obtain the following free-energy correction in the confinement phase

$$\Delta f^{\text{conf}} \simeq -\bar\beta^2/2d-\ln I_0(2\kappa). \tag{3.4}$$

The $\bar\beta$-dependent term is identical to the U(1) result of Alessandrini *et al* (1982). The κ-dependent term can be neglected if the first term dominates. However, when $\bar\beta$ goes to zero and κ is large enough this term dominates. As we see later, this gives the analytical connection between the Higgs and confinement phases for a Higgs charge $q=1$.

In the weak-coupling region we have to deal with the non-trivial saddle points $(m, 0)$ and (m, M). After performing the integration over the fluctuations of the external random fields A_L and B_x we transform the remaining fluctuations in the momentum space $(x \rightarrow p)$ and diagonalise the bilinear operators for the gauge-field fluctuations. The resulting action contains a real and imaginary mode for the Higgs fields. Furthermore it contains $(d-1)$ modes of a degenerate eigenfrequency and one mode which is non-degenerate in both the real and imaginary part of the gauge fluctuations. The imaginary mode of the Higgs field and the imaginary non-degenerate one of the gauge field belong to gauge degrees of freedom. The remaining modes can be integrated out in a unique way. The frequencies corresponding to the gauge degrees of freedom are, for the Higgs fields,

$$A^{\gamma}(p) = \left(\frac{1}{2d}\frac{Z_3}{M} - \frac{Z_2^2}{M^2}\frac{E_{22}E_{11} - E_{21}^2}{E_{11}}\right)\left(\sum_{\mu=0}^{d-1}(1-\cos p_\mu)\right) \tag{3.5a}$$

and for the gauge fields

$$\tilde{\omega}_I(p) = \Xi - \left(\frac{Z_2 E_{21}}{M E_{22}}\right)^2 \sum_{\mu=0}^{d-1}\frac{(1-\cos p_\mu)}{A^{\gamma}(p)} \tag{3.5b}$$

where

$$\Xi = \frac{1}{E_{11}} - \frac{Z_1}{m} \tag{3.5c}$$

is the photon mass. The E_{ij} depend on $f^{(q)}(Z_1, Z_2)$

$$\begin{aligned} E_{ii} &= 1 - (f^{(q)})^{-1}(\partial^2/\partial Z_i^2) f^{(q)} \qquad i = 1, 2 \\ E_{21} &= q_{Z_2/Z_1}(1/Z_2[f^{(q)}]^{-1}(\partial/\partial Z_1) f^{(q)} - E_{22}). \end{aligned} \tag{3.6}$$

Therefore the behaviour of frequencies is connected with the one-link integral.

For saddle points $(m \neq 0,\ M = 0)$ we find

$$f^{(q)} = I_0(Z_1) \qquad Z_2 = 0 \qquad \text{and} \qquad \tilde{\omega}_I(p) = 0 \qquad A^{\gamma}(p) > 0. \tag{3.7}$$

We remark that the A^{γ} correction to the free energy cancels with the remaining Higgs contributions up to a $\ln 2$ term in this case. The zero-frequency $\tilde{\omega}_I(p)$ indicates a U(1) symmetry in the corresponding $(\bar{\beta}, \kappa)$ region (see equation (3.2)). We call this phase a static Coulomb phase, since the Higgs mean value is trivial $(M = 0)$.

In the case of (m, M) the one-link integral can be approximated by (Kasperkovitz 1980)

$$f^{(q)}(Z_1, Z_2) \simeq I_0(Z_1) I_0(Z_2)\left(1 + 2\sum_{k=1}^{\infty}\exp(-t^{(q)}(Z_1, Z_2)k^2)\right) \tag{3.8a}$$

where

$$t^{(q)}(Z_1, Z_2) = \frac{q^2}{2Z_1} + \frac{1}{2Z_2}. \tag{3.8b}$$

The sum in (3.8a) can be neglected if $t^{(q)}$ is large enough. Using $1/d$ as expansion parameter and considering at most $1/d$ terms we find as critical $t^{(q)}$ value

$$t^{(q)}_{\text{crit}}(Z_1, Z_2) = \ln d. \tag{3.9}$$

From this we derive including equation (3.8*b*) and

$$Z_1 \approx 4\bar{\beta} \qquad Z_2 \simeq 2\kappa$$

the following relation between the critical couplings

$$\kappa_{\rm crit} = \frac{\bar{\beta}_{\rm crit}}{4 \ln d} (\bar{\beta}_{\rm crit} - q^2/8 \ln d)^{-1}. \tag{3.10}$$

In the case $t^{(q)} > \ln d$ the sum in equation (3.8*a*) is neglected and the frequencies are

$$\tilde{\omega}_I(p) = 0 \qquad A^{\gamma}(p) = 0. \tag{3.11}$$

The zero frequencies indicate a U(1) symmetry. The existence of a zero frequency for the Higgs field corresponds to the non-trivial Higgs mean value ($M \neq 0$). Therefore we call this region a dynamical Coulomb phase.

If $t^{(q)} < \ln d$ the sum in equation (3.8*a*) is essential. Using an approximation by a Gaussian integral for it we obtain

$$\tilde{\omega}_I(p) = \Xi(1 - 0.29 m^{q-1}) \qquad \Xi = 1/d \frac{Z_3 M q^2}{4 m^{q+1}} \tag{3.12a}$$

$$A^{\gamma}(p) = \frac{1}{2d}\left(1 - \frac{q^2 Z_2}{q^2 Z_2 + Z_1}(1 - m^{q-1})\right) \sum_{\mu=0}^{d-1} (1 - \cos p_\mu). \tag{3.12b}$$

Instead of zero frequencies we find frequencies of order O($1/d$). Since the continuous gauge symmetry is broken we call the corresponding ($\bar{\beta}$, κ) region a Higgs phase. The relation (3.10) between the critical couplings is interpreted as the phase transition line between the Coulomb and Higgs phases. If the sum in equation (3.8*a*) is included in a exact way the frequencies would be continuous functions of the couplings ($\bar{\beta}$, κ). First for increasing couplings they are flat with values near to zero. Then in the region of the critical couplings ($\bar{\beta}_{\rm crit}$, $\kappa_{\rm crit}$) a drastic change in the behaviour can be observed. Finally, these frequencies rapidly increase if the couplings become larger and larger.

The integration over the gauge degrees of freedom is Gaussian in the Higgs phase. In the Coulomb phase we use collective coordinate methods. Following Alessandrini *et al* (1982) we use for the gauge fields a background gauge condition

$$m \sum_{\mu=0}^{d-1} (\xi_L - \xi_{L-\mu}) = 0 \tag{3.13}$$

where ξ_L are the imaginary gauge fluctuations. In the dynamical part we choose a global U(1) symmetry for the Higgs fields

$$\gamma_x = 0 \qquad \forall x \tag{3.14}$$

γ_x denoting the imaginary Higgs fluctuations. The results for the free-energy corrections Δf in the weak coupling region are collected in appendix 1.

4. The phase structure of the lattice Abelian Higgs model

We find two types of phase transitions in the model. The first type describes the transitions from strong to weak coupling. It corresponds to a jump in the mean values of the fields and a free-energy crossing. We determine the phase transition lines by the crossing of the free energies. First we compare the so-called effective free energies containing all numerical significant terms coming from the zeroth-order free energy f_0 and the corrections Δf,

$$f^{(i)}_{\text{eff}}(\bar{\beta}_{\text{eff}}, \kappa_{\text{eff}}) - f^{(j)}_{\text{eff}}(\bar{\beta}_{\text{eff}}, \kappa_{\text{eff}}) = 0. \tag{4.1}$$

Then we correct the results for the couplings using the effective corrections Δf_{eff} following the method described by Alessandrini *et al* (1982). These effective corrections contain only those terms which are apparently proportional to $1/d$. Assuming that the corrections $\delta\bar{\beta}$ and $\delta\kappa$ of the critical couplings are small,

$$\bar{\beta}_{\text{crit}} = \bar{\beta}_{\text{eff}} + \delta\bar{\beta} \qquad \kappa_{\text{crit}} = \kappa_{\text{eff}} + \delta\kappa \tag{4.2}$$

we determine $\delta\bar{\beta}$ and $\delta\kappa$ by

$$\left.\begin{matrix}(\partial/\partial\bar{\beta})(f^{(i)}_{\text{eff}} - f^{(j)}_{\text{eff}})(\bar{\beta}_{\text{eff}}, \kappa_{\text{eff}})\delta\bar{\beta} \\ (\partial/\partial\kappa)(f^{(i)}_{\text{eff}} - f^{(j)}_{\text{eff}})(\bar{\beta}_{\text{eff}}, \kappa_{\text{eff}})\delta\kappa\end{matrix}\right\} = (\Delta f^{(j)}_{\text{eff}} - \Delta f^{(i)}_{\text{eff}})(\bar{\beta}_{\text{eff}}, \kappa_{\text{eff}}). \tag{4.3}$$

The second type of phase transition describes the transition from the Coulomb to the Higgs phases in the weak-coupling region. In the weak-coupling region there is the so-called static Coulomb phase (see § 3). The gauge mean value m is close to one

$$m = 1 - 1/8\bar{\beta}$$

whereas the Higgs mean value M is trivial ($M = 0$). A gauge transformation of the trivial saddle point M leads to the saddle point itself. The frequency corresponding to the gauge degrees of freedom of the gauge field in the loop expansion of the action is zero. Therefore, the effective action is invariant under U(1) gauge transformations.

Furthermore, there is the region of $(\bar{\beta}, \kappa)$ in which the gauge mean value m and the Higgs mean value M are close to one (see equation (3.2)). The frequencies corresponding to the gauge degrees of freedom show a typical behaviour. First, for increasing couplings $(\bar{\beta}, \kappa)$ they are flat with values close to zero. Then, in the region of the critical couplings $(\bar{\beta}_{\text{crit}}, \kappa_{\text{crit}})$ a drastic change in the coupling constant dependence can be observed. Finally, these frequencies rapidly increase if the couplings become larger and larger (see § 3). Note that the mean values m and M as well as the free energy are continuous! Thus, the U(1) gauge symmetry is approximately restored in the $(\bar{\beta}, \kappa)$ region characterised by frequencies near to zero. Therefore, we interpret this region as a dynamical part of the Coulomb phase. The remaining region of coupling constants has non-vanishing frequencies and therefore, the U(1) gauge symmetry is broken. This region is called the Higgs phase. We remark that the κ values in the static part of the Coulomb phase are less than the κ values of the dynamical part of this phase for any fixed value $\bar{\beta}$. Using the approximations described in § 3 (see equations (3.8) and (3.12)) we obtain the relation between the critical couplings of the Coulomb to Higgs phase transition given by equation (3.10).

We have considered the model for Higgs charges $q = 1, 2$ and 6 in $d = 4$ dimensions. First we investigate the confinement to Coulomb transition. We obtain a critical coupling

$$\bar{\beta}_{\text{crit}} = 1.52 \tag{4.4}$$

independent of the Higgs charge q. This result agrees very well with the Monte Carlo prediction by Ranft *et al* (1983). Next we consider the Coulomb to Higgs transition. The transition line is given by equation (3.10). Physically the pole $q^2/(8 \ln d)$ exists only for

$$q^2/(8 \ln d) > \bar{\beta}_{\text{crit}} = 1.52. \tag{4.5}$$

Then a Coulomb phase exists for all values of the Higgs coupling κ up to infinity. In four dimensions it is easy to see that Higgs charges $q \geqslant 5$ satisfy equation (4.2). This is in agreement with the Monte Carlo calculations (Ranft *et al* 1983).

Finally we discuss the confinement to Higgs transition for a Higgs charge $q = 1$. For small enough $\bar{\beta}$ the effective free energies of the two phases are given by

$$f_{\text{eff}}^{\text{Higgs}} \simeq -\bar{\beta} - \ln I_0(2\kappa M^2) \qquad f_{\text{eff}}^{\text{conf}} = -\ln I_0(2\kappa). \tag{4.6}$$

For κ large enough and M close to one we find for $\bar{\beta} \leqslant 1$ that the coupling constant dependence of these free energies is approximately the same. Therefore we can regard the two phases as analytically connected in this region. By this argument the value $\bar{\beta} = 1$ is used as the upper limit for the analytically connected region. Including the effective corrections we obtain an end point $(\bar{\beta}^*, \kappa^*)$ of the confinement to the Higgs transition

$$(\bar{\beta}^*, \kappa^*) = (1.125, 0.36) \tag{4.7}$$

which is in a good agreement with the previous data (Ranft *et al* 1983).

Table 1. The results for all phase transitions calculated for dimension $d = 4$ and Higgs charges $q = 1$, 2 and 6.

q	Confinement-Higgs transition	Confinement-Coulomb transition	Coulomb-Higgs transition
1	End-point: $\bar{\beta}^* = 1.125$ $1.52 \geqslant \bar{\beta}_{\text{crit}} \geqslant \bar{\beta}^*$ $\kappa_{\text{crit}} = 0.49 - 0.12\bar{\beta}_{\text{crit}}$	$\bar{\beta}_{\text{crit}} = 1.52$	$q^2/(8 \ln d) = 0.09$
2	$\kappa_{\text{crit}} = \frac{1}{4}\bar{\beta}_{\text{crit}}(\bar{\beta}_{\text{crit}} - \ln 2)^{-1} - 0.16$	$\bar{\beta}_{\text{crit}} = 1.52$	$q^2/(8 \ln d) = 0.36$
6	—	$\bar{\beta}_{\text{crit}} = 1.52$	$q^2/(8 \ln d) = 3.25$

In table 1 we summarise the results for all phase transitions calculated for dimensions $d = 4$ and Higgs charges $q = 1$, 2 and 6. Our results for the improved mean-field calculation (IMF) compared with the Monte Carlo data (MC) and the results of the mean-field tree approximation (MF) by Ranft *et al* (1983) are presented in figures 1-3. We present the phase transitions in the plane of the couplings $\bar{\beta}$ and κ for Higgs charges $q = 1$, 2 and 6. Note that our definitions for the couplings differ from those in the paper of Ranft *et al* (1983). The chain curves denote the phase transitions obtained by MF, the full ones are phase transitions resulting from IMF. The MC data are represented by open circles. The bends in the transition lines are due to different analytical approximations. These approximations correspond to the two types of phase transitions described above. The Coulomb to Higgs transition is described by equation (3.2). The strong to weak coupling transitions (confinement to Coulomb phases, confinement to Higgs phases) are considered separately. The results are the analytical approximations for the phase transition lines shown in table 1.

In the case of the Higgs charge $q = 1$ (figure 1) the end point of the confinement to Higgs transition in the IMF calculation is in good agreement with the Monte Carlo results. Figure 2 shows the phase structure for a Higgs charge $q = 2$. Both MF and

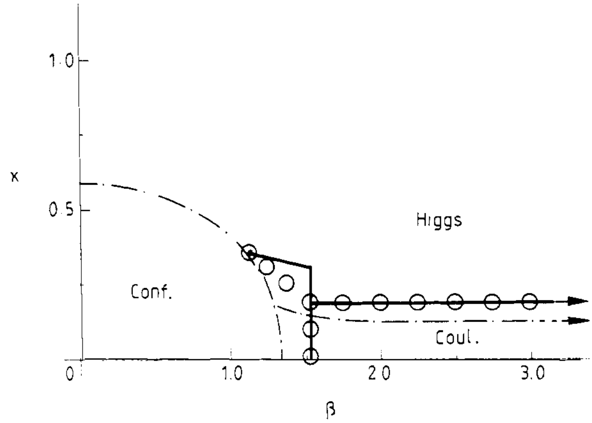

Figure 1. The phase structure of the model for Higgs charge $q = 1$ and dimension $d = 4$ in the $(\bar{\beta}, \kappa)$ coupling constant plane. ○ ○, Monte Carlo (MC); —·—, mean-field tree approximation (MF) $q = 1$; ——, improved mean field (IMF).

IMF results are in reasonable agreement with the data. In figure 3 we represent our results for the Higgs charge $q = 6$. Using the improved mean-field calculation we can qualitatively reproduce the correct phase structure as predicted by Monte Carlo simulations. For increasing values of coupling κ the Coulomb phase remains.

5. Conclusions and summary

We have found that the improved mean-field calculation including corrections on the one-loop level leads to the true phase structure of the lattice Abelian Higgs model as found in Monte Carlo studies for all Higgs charges considered. We emphasise that to distinguish the phases of the model we use the different behaviour of the saddle-point

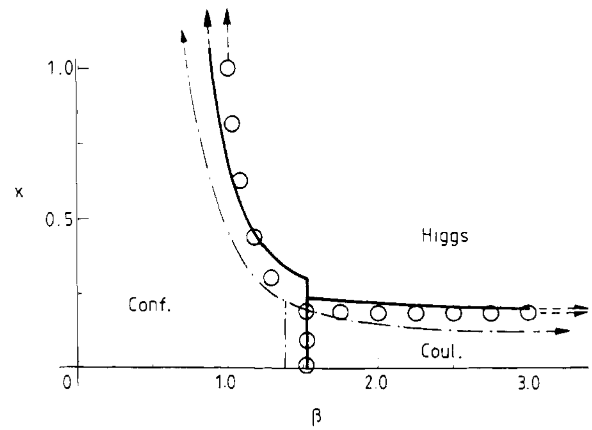

Figure 2. The phase structure for Higgs charge $q = 2$ and dimension $d = 4$. Symbols as in figure 1.

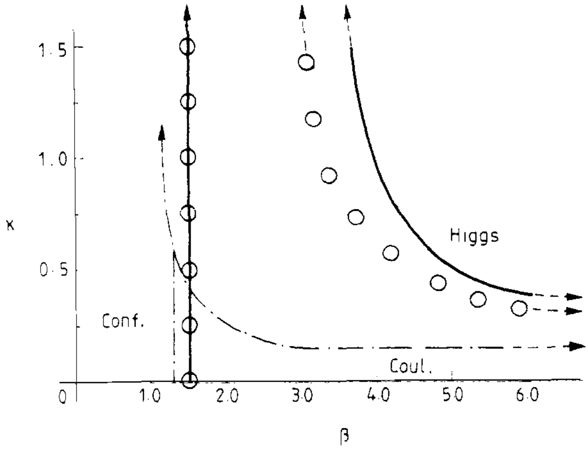

Figure 3. The phase structure for Higgs charge $q = 6$ and dimension $d = 4$. Symbols as in figure 1.

values and the effective action under gauge transformation. The confinement phase can be characterised by a trivial U(1) symmetry. This means that a gauge transformation of the trivial saddle points ($m = 0$, $M = 0$) leads to the saddle points themselves. The characteristic feature of the Coulomb phase is the non-trivial U(1) symmetry. The gauge transformed non-trivial saddle points ($m \neq 0$, $M \neq 0$) differ from the original ones by a phase, whereas the effective action is invariant under this transformation. In the Higgs phase the continuous gauge symmetry is broken so that the effective action is not invariant under the U(1) transformation. For the transitions from the confinement to the Higgs or Coulomb phases, corresponding to a strong to weak coupling transition, at least one saddle point becomes non-trivial. Thus the jumps in the mean values of the fields corresponding to a free-energy crossing are used as a phase transition criterion. From the mean-field equations it follows that the mean values of the fields behave as first derivatives of the free energy. Therefore we can interpret the transition with jumps in the mean values as first-order phase transitions. In the weak-coupling region the behaviour of the effective action under gauge transformation is connected with the coupling constant dependence of the frequencies in the loop expansion. The frequencies corresponding to gauge degrees of freedom are zero in the Coulomb phase and increase rapidly approaching the Higgs phase. The drastic change in the frequency behaviour corresponds to continuous free energy and continuous mean values for gauge and Higgs fields. Therefore, we interpret this transition as a second-order one.

The phase structure obtained is in qualitative agreement with the Monte Carlo simulations (Ranft *et al* 1983). A qualitative change of the phase structure due to corrections beyond the one-loop level seems to be impossible in view of the magnitude of these corrections by Flyvbjerg *et al* (1983) and Flyvbjerg (1984). Therefore, we have calculated the corrections to the free energy per link up to one loop. The pure saddle-point approximation does not contain all essential terms of the free energy. As a consequence the tree approximation does not reproduce the end point in the confinement to Higgs transition for the Higgs charge $q = 1$ as found including the one-loop corrections. In the weak-coupling region we use the frequency behaviour for the determination of the phase transition lines. We find for a Higgs charge $q = 6$

that with increasing Higgs coupling the Coulomb phase remains in a region of non-trivial Higgs mean value ($M \neq 0$). In the tree approximation one characterises the phases by the mean values of the fields so that there is only the phase transition criterion explained first. Therefore, one obtains only the static part of the Coulomb phase where the Higgs mean value is trivial ($M = 0$). The dynamical part with non-trivial Higgs mean value is interpreted as a part of the Higgs phase. Therefore, the tree approximation can not reproduce the Coulomb phase for increasing Higgs coupling in the case of a Higgs charge $q = 6$.

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Appendix 1

For the static (stat) and dynamical (dyn) part of the Coulomb phase the free energy corrections are

$$\Delta f^{\text{Coul}}_{\text{stat}} = \frac{1}{d}\left[-\tfrac{1}{2}\ln(4\pi d m^2) - \tfrac{1}{2}\ln\left(\frac{Z_1}{2m - Z_1 + Z_1 m^2}\right) - \ln 2 + \tfrac{3}{8} - \tfrac{1}{2}(-D + \tfrac{1}{2}D^2)\right]$$

$$\Delta f^{\text{Coul}}_{\text{dyn}} = \Delta f^{\text{Coul}}_{\text{stat}} + \frac{1}{d}\left[-\tfrac{1}{2}\ln M^2 + \left(1 - \frac{2}{N}\right)\tfrac{1}{2}\ln 2\pi \right.$$
$$\left. - \tfrac{1}{2}\ln\left(\frac{Z_3}{2M - Z_3 + Z_3 M^2}\right) - \tfrac{1}{2}(-L + \tfrac{1}{2}L^2)\right]$$

$$D = \frac{1}{2m}(Z_1 - m - Z_1 m^2) \qquad L = \frac{1}{m^{2q}}[2m^{2q} - Z_2(1 - m^{2q})]\frac{1}{2M}(Z_3 - M - Z_3 M^2)$$

and in the Higgs phase we find

$$\Delta f^{\text{Higgs}} = \tfrac{1}{2}\ln\left[\left(\frac{2Z_1 + q^2 Z_2}{2(Z_1 + q^2 Z_2)}\right)\left(\frac{m}{Z_1 + q^2 Z_2} - \frac{Z_1(1 - m^q)}{2Z_1 + q^2 Z_2}\right)\frac{Z_1}{m}(1 + 4m\Xi)\right]$$
$$+ \frac{1}{2d}\left[-\ln\left(\frac{Z_1}{m}\right) - \ln 2 - \ln\left(\frac{2M}{2M - Z_3 + Z_3 M^2}\right)\right.$$
$$\left. + \ln\left(\frac{q^2 m^{q-1} Z_2 + Z_1}{q^2 Z_2 + Z_1}\right) - (-D + \tfrac{1}{2}D^2) - (-L + \tfrac{1}{2}L^2)\right]$$

$$D = \frac{2Z_1}{2Z_1 + q^2 Z_2}\,\frac{(Z_1 + q^2 Z_2)(1 - m^2) - m}{2m}$$

$$L = \left(\frac{2q^2 Z_2 + Z_1}{q^2 Z_2 + Z_1} - \frac{Z_2(1 - m^{2q})}{m^{2q}}\right)\frac{Z_3 - M - Z_3 M^2}{2M}$$

Appendix 2. The $(\kappa-\kappa_c)^{1/2}$ behaviour of $M(\kappa)$

Using the mean-field equations (3.1) we find a Higgs mean value

$$M = I_1(z_3)/I_0(z_3) \qquad \text{and} \qquad z_3 = 4d\kappa m^q M \tag{A2.1}$$

where the $I_n(z_3)$ are modified Bessel functions. In the Higgs phase we expect a Higgs mean value M near to one and a very large value for z_3. Therefore the modified Bessel functions $I_n(z)$ can be approximated by (Kasperkovitz 1980)

$$I_n(z_3) \simeq \frac{e^{z_3}}{(2\pi z_3)^{1/2}} \exp(-n^2/2z_3). \tag{A2.2}$$

Using the approximation (A2.2) we find a Higgs mean value

$$M \simeq \exp(-1/2z_3). \tag{A2.3}$$

We can expand the exponential function in equation (A2.3) for large enough values of z_3 and obtain

$$M \simeq 1 - 1/2z_3. \tag{A2.4}$$

Equation (A2.4) is consistent with the assumption of a value of M close to one in the Higgs phase. The gauge mean value m behaves as

$$m \leqslant 1 \tag{A2.5}$$

in both the Coulomb and the Higgs phases. Therefore we can approximate

$$z_3 \simeq 4d\kappa.$$

The Higgs mean value is then given by

$$M \simeq 1 - 1/8d\kappa \tag{A2.6}$$

as in equation (3.2). Note that the approximation (A2.3) works also for $z_3 \to 0$ and $M \to 0$! It shows at least the true qualitative behaviour of M. Due to the gauge mean value m being close to one the dependence of the Higgs mean value M on the gauge coupling $\bar{\beta}$ is not essential for the Coulomb to Higgs phase transition. We regard M and z_3 as functions of the Higgs coupling κ whereas $\bar{\beta}$ is a parameter,

$$\begin{aligned} M = M(\kappa) \qquad z_3 &= CM(\kappa)\kappa \qquad C = 4dm^q \simeq \text{constant} \\ &= z_3(\kappa) \qquad \text{for fixed } d, q. \end{aligned} \tag{A2.7}$$

The Higgs mean value $M(\kappa)$ shows a drastic change in the behaviour going from the Higgs to the Coulomb phases. Therefore we can assume that the approximation (A2.3) remains valid also for Higgs couplings

$$\kappa \gtrsim \kappa_c.$$

where κ_c denotes the critical coupling of the phase transition. An expansion of the Higgs mean value $M(\kappa+\Delta\kappa)$ given by equation (A2.3) in terms of the small $\Delta\kappa$ yields

$$M(\kappa+\Delta\kappa) \simeq \exp\left(-\frac{1}{2z_3(\kappa)}\right)\left(1 + \frac{1}{2z_3^2(\kappa)} \left.\frac{dz_3}{d\kappa}\right|_\kappa \Delta\kappa\right). \tag{A2.8}$$

For $\Delta\kappa = \kappa_c - \kappa$ and using $M(\kappa_c) \equiv 0$ we obtain

$$M(\kappa_c) \equiv 0 = \exp\left(-\frac{1}{2z_3(\kappa)}\right)\left(1 - \frac{1}{2z_3^2(\kappa)} \left.\frac{dz_3}{d\kappa}\right|_\kappa (\kappa - \kappa_c)\right)$$

or

$$z_3^2(\kappa) = \tfrac{1}{2} \left.\frac{dz_3}{d\kappa}\right|_\kappa (\kappa - \kappa_c). \qquad \text{(A2.9)}$$

The consideration of equation (A2.7) for z_3 yields a Higgs mean value

$$M(\kappa) = \frac{1}{\sqrt{2}C\kappa}\left(\left.\frac{dz_3}{d\kappa}\right|_\kappa\right)^{1/2} (\kappa - \kappa_c)^{1/2}. \qquad \text{(A2.10)}$$

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